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The Computational Stability Properties of the Shuman Pressure Gradient Averaging Technique

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The stability properties of the Shuman pressure gradient averaging technique are investigated with the linearized shallow water equations. In the simplest case an analytic expression is obtained for the stability condition. The maximum time step is twice the value for the leapfrog scheme. When a mean flog is added to the equations the time step must be reduced. The time averaging suggested by Robert is examined and again leads to a shorter time step. In each case however, the use of the Shuman averaging allows a significantly longer time step.

1. INTRODUCTION

Shuman [5] has proposed a modification to the leapfrog scheme for the primitive equations that are used in numerical weather prediction. The scheme that uses a weighted average of the pressure gradient at the past, present, and future times, allows a longer time step than the one given by the CFL condition. Shuman, Brown, and Campana [6] have carried out the linear analysis of this modification for the three cases: (1) the barotropic model, (2) a two-layer model with pressure as the vertical coordinate, and (3) a two-layer model with the Phillips [3] sigma coordinate system. In this paper we will extend and amplify their analysis of the linearized barotropic model to include the physically important class of problems where the basic model is modified by inclusion of a mean flow term. We will also examine the effect of including the time filter designed by Robert [4], both with and without a mean flow present. It will be seen that both the presence of a mean flow and the time filtering reduce the time step, but that the use of the Shuman technique is still advantageous.

2. Basic Solution

The linearized equations for the barotropic model with no Coriolis force and no mean flow are:

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x}, \tag{1}$$

$$\frac{\partial \phi}{\partial t} = -\Phi \frac{\partial u}{\partial x}. \tag{2}$$

Here, u is the velocity component in the x-direction and ϕ/g is the departure of the free surface height from its mean value Φ/g. These equations describe shallow water waves that move with speed $\Phi^{1/2}$. Shuman, Brown, and Campana [6] numerically determined the stability curve for this problem, with pressure gradient averaging. We will show that their results are analytically derivable.

The finite difference approximations to equations (1) and (2) with the Shuman pressure gradient averaging included are:

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + \frac{1}{2\Delta x}[\phi_{j+1}^n - \phi_{j-1}^n + \alpha[\phi_{j+1}^{n+1} - \phi_{j-1}^{n+1} + \phi_{j+1}^{n-1} - \phi_{j-1}^{n-1} - 2(\phi_{j+1}^n - \phi_{j-1}^n)]] = 0, \tag{3}$$

$$\frac{\phi_j^{n+1} - \phi_j^{n-1}}{2\Delta t} + \Phi\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right) = 0, \tag{4}$$

where the discretization uses $x = j\,\Delta x$ and $t = n\,\Delta t$. The usual leapfrog differencing is obtained by setting $\alpha = 0$. This scheme is explicit since ϕ^{n+1} may be obtained from (4) before it is needed in (3).

To obtain the computational stability properties of this scheme, we substitute the following expressions

$$\begin{aligned} u_j^n &= A\omega^n \exp(ik\,\Delta x j), \\ \phi_j^n &= B\omega^n \exp(ik\,\Delta x j), \end{aligned} \tag{5}$$

into (3) and (4). After the constants A and B have been eliminated we obtain the following quartic equation for ω:

$$\omega^4 + 4S\alpha\omega^3 - 2[1 - 2S(1 - 2\alpha)]\,\omega^2 + 4S\alpha\omega + 1 = 0, \tag{6}$$

where

$$S \equiv (\Delta t/\Delta x)^2\,\Phi \sin^2 k\,\Delta x. \tag{7}$$

Since S and α are real, the roots of (6) are either real or in complex conjugate pairs. Due to the symmetry in the differencing we expect that $|\omega| = 1$ for some range of the parameters. As a result the polynomial factors to

$$(\omega^2 + 2a_1\omega + 1)(\omega^2 + 2a_2\omega + 1) = 0. \tag{8}$$

If we expand (8) and compare with (6) we find that a_1 and a_2 must satisfy the following equation:

$$a_i^2 - 2S\alpha a_i + S(1 - 2\alpha) - 1 = 0, \qquad i = 1, 2. \tag{9}$$

The solutions to this equation are

$$a_i = S\alpha \pm ((S\alpha + 1)^2 - S)^{1/2}. \tag{10}$$

It can be seen from (8) that only when the a_i's are real and of magnitude less than or equal to unity will all the roots of (8) have $|\omega| = 1$. The value of a_i will be real when the quantity under the radical is nonnegative; therefore, this condition can be written

$$(S\alpha + 1)^2 - S \geqslant 0. \tag{11}$$

If we choose the equality we will obtain the following stability relation:

$$S = \frac{1 - 2\alpha - (1 - 4\alpha)^{1/2}}{2\alpha^2}, \tag{12}$$

where the minus sign in front of the radical was chosen to make the expression reasonable in the limit as $\alpha \to 0$. The condition that the a_i's have magnitude not greater than 1 leads to the condition

$$\alpha \leqslant \tfrac{1}{4}. \tag{13}$$

This is consistent with (12) which becomes complex for $\alpha > \frac{1}{4}$.

The combination of conditions (12) and (13) is given in Fig. 1 as the curve labeled $\sigma = 0$. Shuman, Brown, and Campana [6] determined essentially this curve from their numerical solution for the roots of (6). The maximum value of S which is $S = 4$, occurs at $\alpha = \frac{1}{4}$. For the usual leapfrog differencing ($\alpha = 0$) the maximum value of S that allows computationally stable solutions is $S = 1$. Since S is proportional to Δt^2 (see Eq. (7)) it follows that the use of the Shuman pressure gradient averaging in this linear system allows a doubling of the time step as compared with the standard leapfrog scheme. However, it may be difficult to achieve this factor

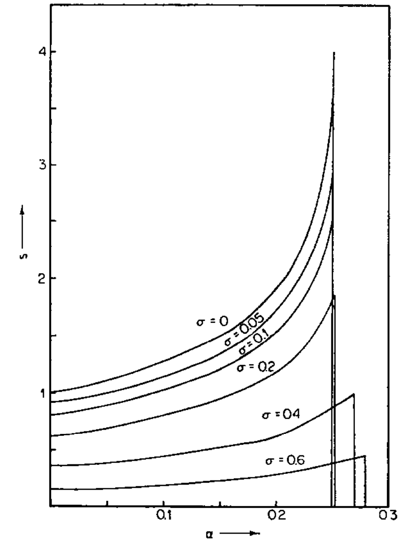

FIG. 1. Computational stability curves for selected values of σ.

of 2 in practice when other effects are included since the width of the stable region goes to zero as S approaches 4. In fact a value of α which is slightly less than $\frac{1}{4}$ would probably be preferable.

3. SOLUTIONS WITH MEAN FLOW

In this section we add the effects of a constant mean flow to the stability analysis. The linearized equations are:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial x}, \tag{14}$$

$$\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x} = -\Phi\frac{\partial u}{\partial x}. \tag{15}$$

Normally, the addition of the mean flow terms does not have a great effect on the computational stability criteria since the phase speed of the external gravity waves $\Phi^{1/2}$, is generally much greater than the speed of the mean wind, U.

When Eqs. (14) and (15) are put in finite difference form with the use of the approximations in (3) and (4), the mean advection terms are evaluated with centered finite differences, and the relations (5) are substituted into the finite difference forms of (14) and (15), we obtain the following equation for ω:

$$\omega^4 + 4(S\alpha + i\sigma)\,\omega^3 + 2[2S(1-2\alpha) - (1+2\sigma^2)]\,\omega^2 + 4(S\alpha - i\sigma)\,\omega + 1 = 0, \tag{16}$$

where

$$\sigma = (U\,\Delta t/\Delta x)\sin k\,\Delta x. \tag{17}$$

Note that dividing (16) by ω^4 and taking the complex conjugate of the equation, yields an equation in $1/\omega^*$ with the same coefficients as (16). Thus, it follows that ω is a root of (16) if and only if $(\omega^*)^{-1}$ is also a root, and therefore, the roots must be distributed in exactly one of the following ways:

Case I. Four roots (including multiple roots) on the unit circle

$$(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}, e^{i\phi_4};\ \phi_1 + \phi_2 + \phi_3 = -\phi_4).$$

Case II. Two roots on the unit circle, two roots off the unit circle

$$(e^{i\phi_2}, e^{i\phi_3}, \rho e^{i\phi_1}, (1/\rho)\,e^{i\phi_1};\ 2\phi_1 + \phi_2 = -\phi_3).$$

Case III. Four roots off the unit circle

$$(\rho e^{i\phi}, (1/\rho)\,e^{i\phi}, \zeta e^{-i\phi}, (1/\zeta)\,e^{-i\phi}, \rho, \zeta \neq 1).$$

In each case the fact that the product of the roots is equal to the last term in (16) was used.

The analysis of Cases I–III indicates that (16) has at least one factorization of the form

$$(\omega^2 + 2ae^{i\theta/2}\omega + e^{i\theta})(\omega^2 + 2be^{-i\theta/2}\omega + e^{-i\theta}) = 0, \tag{18}$$

for some angle θ, and a and b real. Furthermore it is easily shown, by analyzing the possible combinations, that in either of the unstable cases (Cases II or III), the factorization (18) must be unique and either a or b (or both) greater than unity in magnitude, whereas the stable case will generally have three distinct representations with both a and b less than or equal to unity in magnitude. The only exception will be when stable, but degenerate (multiple) roots occur.

If we expand (18) and compare with (16) we can obtain the following equations:

$$a = S\alpha \sec\theta/2 + \sigma \csc\theta/2, \tag{19}$$

$$b = S\alpha \sec\theta/2 - \sigma \csc\theta/2, \tag{20}$$

$$S^2\alpha^2 \sec^2\theta/2 - S + S\alpha + \cos^2\theta/2 - \sigma^2 \cot^2\theta/2 = 0. \tag{21}$$

When we write $u = \cos^2 \theta/2$ and use trigonometric identities we can obtain the following polynomial in u:

$$P_\sigma(u) \equiv (u - 1)(u^2 + S(2\alpha - 1)\, u + S^2\alpha^2) + \sigma^2 u^2 = 0, \tag{22}$$

where one different factorization of the form (18) arises from each distinct solution of (22) in $0 \leqslant u \leqslant 1$. Thus by our above comments a sufficient condition for stability is that (22) have more than one solution in [0, 1]. The appearance of (22) as a cubic should be expected, since as noted above in the generalized case of stability, we expect three distinct factorizations. When only one solution of (22) occurs in [0, 1], then there must be instability unless (16) has a triple root. It can be shown that there is at most one value of σ for which (16) has a triple root.

With this characterization consider the qualitative behavior of $P_\sigma(u)$. Note for all $u > 0$, $\sigma > 0$ that $P_\sigma(u)$ will lie above the curve

$$P_0(u) = (u - 1)(u^2 + S(2\alpha - 1)\, u + S^2\alpha^2), \tag{23}$$

where it is easily seen that

$$P_0(0) = P_\sigma(0) = -S^2\alpha^2, \tag{24}$$

$$P_0(1) = 0, \tag{25}$$

$$P_0'(1) = (S\alpha + 1)^2 - S. \tag{26}$$

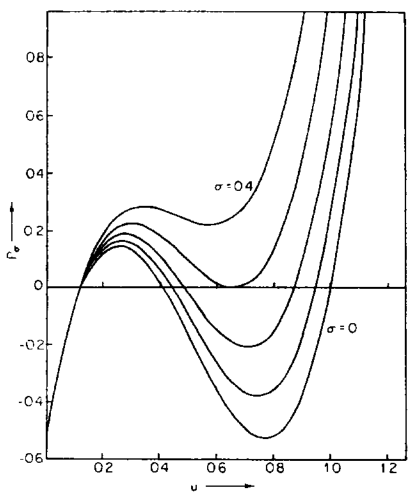

FIG. 2. The function $P_\sigma(u)$ for $S = 1.0$ and $\alpha = 0.23$ for selected values of σ.

Also note that for fixed u, $P_\sigma(u)$ is a strictly increasing function of σ. These observations now allow us to visualize fairly easily how an instability develops. Consider Fig. 2, which shows a typical situation when $P_0(u)$ has three distinct root $0 \leqslant u \leqslant 1$, i.e., $\sigma = 0$ yields a stable procedure. Then observe that as σ increases the curve moves upward, causing the value of the smaller root to decrease, and the two larger roots to move closer together. Eventually, for some value σ_{max}, the larger two roots degenerate to a single double root with $P_{\sigma_{max}}(u)$ tangent to the axis there. Finally for $\sigma > \sigma_{max}$, the curve detaches from the axis, leaving only the single smaller root, and hence instability must occur.

Intriguingly, this construction also allows us to visualize a situation when $\sigma > 0$ will yield a stable procedure even though $\sigma = 0$ does not. This is shown in Fig. 3, where as σ increases the curve first attaches itself producing a double lower root,

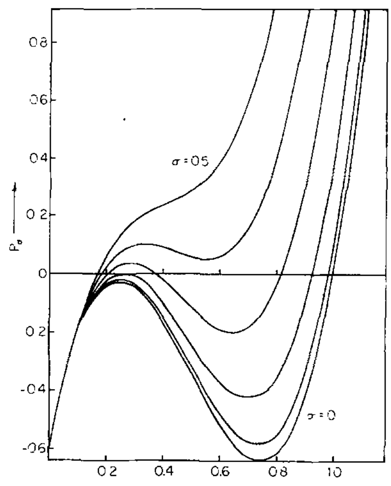

FIG. 3. The function $P_\sigma(u)$ for $S = 1.0$ and $\alpha = 0.25326$ for selected values of σ.

which then moves apart, until eventually a double upper root appears, followed by instability. Numerical solution of (16) verifies that this in fact occurs, as evidenced by the curves for $\alpha > \frac{1}{4}$ in Fig. 1.

Now consider the situation when $P_0'(1) > 0$ and $0 \leqslant \alpha \leqslant \frac{1}{4}$. The roots of $P_0(u) = 0$ are

$$u_0 = 1, \tag{27}$$

$$u_1 = (S/2)[(1 - 2\alpha) - (1 - 4\alpha)^{1/2}], \tag{28}$$

$$u_2 = (S/2)[(1 - 2\alpha) + (1 - 4\alpha)^{1/2}]. \tag{29}$$

Since $(1 - 4\alpha) \leqslant (1 - 2\alpha)^2$, it follows that $P_0(u)$ has a shape as indicated in Fig. 2. The maximum stable value of σ (hereafter called $\sigma_{\max}$) occurs when the larger root of $P_\sigma'(u) = 0$ is also a double root of $P_0(u)$.

Since $P_\sigma(u)$ is a cubic, the conditions can be worked our formally; however, the resulting conditions are too cumbersome to provide computational or analytic insights. Therefore, we shall present some simpler necessary conditions and approximate expressions.

First, observe that if $P_0'(u) \leqslant 0$, $P_0(u)$ has exactly one root in $0 < u < 1$, hence, $P_\sigma(u)$ must have exactly one root in $0 \leqslant u \leqslant 1$. Thus $P_0'(u) > 0$ is a necessary condition for stability when $\sigma > 0$. (Observe the condition that (26) be nonnegative is identical to (11).)

A rather accurate approximation to $\sigma_{\max}$ for this situation can be obtained by noting that the onset of instability must correspond to $a = 1$, since $a > 1$ yields, from (18), a solution with $|\omega| > 1$. Thus, from (19) $\sigma_{\max}$ satisfies

$$S\alpha \sec \theta/2 + \sigma_{\max} \csc \theta/2 = 1,$$

or

$$\frac{S\alpha}{(u_L)^{1/2}} + \frac{\sigma_{\max}}{(1 - u_L)^{1/2}} = 1, \tag{30}$$

where u_L denotes the smallest root of $P_{\sigma_{\max}}(u)$. This is necessary since both a and b must be continuously dependent on σ.

The u_L in (30) is not known exactly, but the examination of Fig. 2 indicates that

$$u_L \sim u_1 , \tag{31}$$

where u_1 is the lower root of $P_0(u)$. Thus, to a first approximation, stability will occur for

$$0 \leqslant \sigma_{\max} \cong (1 - u_1)^{1/2}\{1 - (S\alpha/(u_1)^{1/2})\}, \qquad 0 < \alpha < (1/4), \tag{32}$$

where u_1 is given by (28). It will be seen that this equation gives an excellent approximation to the values obtained numerically. Unfortunately, a similar approximation for $\frac{1}{4} < \alpha$ has not yet been found.

The roots of (16) were computed numerically and the resulting stability curves are given in Fig. 1 for selected values of σ. All the curves in Fig. 1 have similar shapes with the lowest curves corresponding to the highest values of σ. The following stability condition for $\alpha = 0$ is easily obtained:

$$S^{1/2} + |\sigma| \leqslant 1. \tag{33}$$

For $\sigma = 0.1$, the increase in Δt over the value for $\alpha = 0$ (see Eq. (17)) is about

80 %. A typical value of σ for operational numerical prediction models would be between 0.1 and 0.2 Figure 1 also shows that stability can occur for the larger values of σ in the region $\alpha > \frac{1}{4}$. However, these values of α should not be used because there will always be a value of $k\,\Delta x$ in (17) that will give an arbitrarily small value of σ and therefore instability.

Tables I and II compare values on the numerical stability curve from Fig. 1 with values computed from (30). The agreement is quite good with the largest difference occurring for α near $\frac{1}{4}$ and for larger values of σ.

TABLE I

Comparison of the values of S[a]

α	0	0.1	0.2	0.225	0.25
S_{an}	0.78	1.00	1.50	1.75	2.62
S_{num}	0.80	1.02	1.55	1.80	2.50

[a] On the stability curve from Fig. 1 (S_{num}) with the values of S computed from Eq. (30) (S_{an}) for $\sigma = 0.1$.

TABLE II

Comparison of the Values of S[a]

α	0	0.1	0.2	0.225	0.25
S_{an}	0.33	0.43	0.61	0.71	1.04
S_{num}	0.35	0.46	0.62	0.74	0.88

[a] Same as Table I except for $\sigma = 0.4$.

4. Solution with Time Filter

Time filtering has been used in numerical weather prediction models to damp both physical and numerical noise (Robert [4], Haltiner and McCollough [2]). Consider the following centered time filter:

$$\bar{F}(t) = F(t) + \gamma[F(t + \Delta t) + \bar{F}(t - \Delta t) - 2F(t)]. \tag{34}$$

This form is convenient for operational prediction because it uses the previous averaged value, which saves machine storage. When this filter is used in linear equations which have solutions proportional to ω^n, Eq. (34) takes the form

$$\bar{F}(t) = (F(t) + \gamma[F(t + \Delta t) - 2F(t)]/(1 - \gamma\omega^{-1}). \tag{35}$$

Here we have used the relation $\bar{F}(t - \Delta t) = \omega^{-1}\bar{F}(t)$.

This time filter is introduced into the finite difference Eqs. (3) and (4) by replacing u_j^{n-1} and ϕ_j^{n-1} with the filtered values obtained from (35). When the relations (5) are introduced into the time averaged difference equations we obtain the following:

$$\omega^4 + 4[S\alpha - \gamma]\,\omega^3 + [4(\gamma(1 + \gamma) + S[1 - 2\alpha(1 + \gamma)]) - 2]\,\omega^2 + 4[S(\alpha(1 + 2\gamma) - 2\gamma) + \gamma(1 - 2\gamma)]\,\omega + 4S\gamma(\gamma - \alpha) + (1 - 2\gamma)^2 = 0. \tag{36}$$

If we consider the special case of no pressure gradient averaging ($\alpha = 0$), the equation reduces to

$$[\omega^2 - 2\gamma\omega - (1 - 2\gamma)]^2 = -4S(\omega - \gamma)^2. \tag{37}$$

Take the square root of both sides of this equation and solve for ω which yields

$$\omega = \gamma \pm iS^{1/2} \pm ((\gamma - 1)^2 - S)^{1/2}. \tag{38}$$

This result was obtained by Asselin [1] who has discussed the solutions in detail. When S is sufficiently small the solutions will be damped. The critical value of S, which is always less than 1, decreases as γ increases. However, in the damping region, the damping rate increases with increasing γ.

In the general case the roots to (36) must be found numerically. Figure 4 contains the curves that separate the unstable solutions from the stable solutions for selected values of γ. The left-hand limits of the curves show the reduction in the critical S as a function of γ for $\alpha = 0$. The curve for $\gamma = 0.05$ closely approximates the curve for $\sigma = 0$ in Fig. 1. As γ increases the maximum stable value decreases and shifts to the right. In fact sizable stable regions exist for $\alpha > \frac{1}{4}$ depending on γ. For $\gamma = 0$ there are no stable solutions for $\alpha > \frac{1}{4}$. J. A. Brown, Jr. has obtained similar results independently (private communication).

We now consider the effect of the time averaging on the solutions when the mean flow is included. When the time averaging effects are added to equation (16) we obtain:

$$\omega^4 + 4[S\alpha - \gamma + i\sigma]\,\omega^3 + [4(\gamma(\gamma + 1) - \sigma^2 + S(1 - 2\alpha(\gamma + 1))) - 2 - 12\sigma\gamma i]\,\omega^2 + 4[S(\alpha(1 + 2\gamma) - 2\gamma) + \gamma(1 - 2\gamma + 2\gamma^2) + i\sigma(2\gamma(\gamma + 1) - 1)]\,\omega + 4S\gamma(\gamma(1 - \sigma^2) - \alpha) + (1 - 2\gamma)^2 + i4\sigma\gamma(1 - 2\gamma) = 0. \tag{39}$$

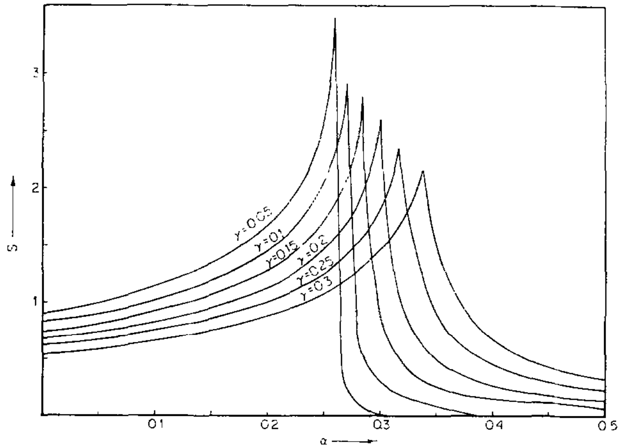

FIG. 4. Computational stability curves for selected values of γ for $\sigma = 0$.

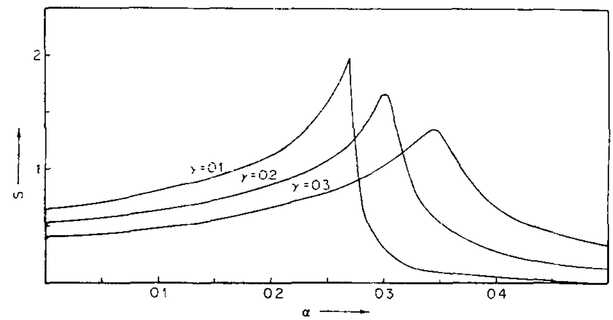

FIG. 5. Computational stability curves for selected values of γ for $\sigma = 0.1$.

Figure 5 contains the stability curves which were obtained by numerical solution of (39) for the value $\sigma = 0.1$. The curves which are for $\gamma = 0.1, 0.2, 0.3$ show smaller values of maximum values of S than are seen in Fig. 3. However, the starting points ($\alpha = 0$) are also smaller. In general the peaks are located at about the same values of α and the peaks are broader. Both Figs. 4 and 5 show a large improvement in maximum S over the value with no pressure gradient averaging ($\alpha = 0$).

5. CONCLUSIONS

The results of Shuman, Brown, and Campana for the stability properties of the Shuman [5] pressure gradient averaging technique with the linearized shallow water equations have been extended to include problems characterized by a persistent mean flow. In addition, a numerical result of Shuman, Brown, and Campana has been derived analytically in Section 2, showing that in the simplest case the time step can be doubled for $\alpha = \frac{1}{4}$. However, since the width of the stable region becomes very narrow as $\alpha = \frac{1}{4}$ is approached, the best value of α would be slightly less than $\frac{1}{4}$. We have demonstrated that, when a mean flow is included the time step must be reduced; however, for reasonable values of the mean flow ($\sigma = 0.1$ to 0.2) the time step can still be increased by 70 to 80%. The time averaging of all variables which was suggested by Robert [4] has been used to damp unwanted high frequency components in numerical forecasts. The use of the time filtering, however, requires a smaller time step, and therefore, more computation time. When the time filtering is used in conjunction with the pressure gradient averaging, the time step can be significantly increased although for the larger values of γ the time step may not be much larger than with no time or pressure gradient averaging. When the time averaging is used the optimum value of α is critically dependent on γ. The addition of the mean flow decreases the time step, but does not appreciably affect the optimal α.

The Shuman [5] pressure gradient averaging technique has been used operationally at the National Meteorological Center and it is now undergoing tests at the Fleet Numerical Weather Central. This technique should be useful in other fluid dynamical applications provided that the velocities are appreciably less than the fastest gravity waves.

Acknowledgments

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